Adaptive NN-Based Reference-Tracking Control of Uncertain Nonlinear Parabolic PDE Systems

Jingting Zhang, Yan Gu, Paolo Stegagno, Wei Zeng, Chengzhi Yuan

Abstract— **This paper is focused on the reference-tracking control problem of distributed parameter systems modeled by a class of parabolic partial differential equations (PDEs) with uncertain nonlinear dynamics. An adaptive tracking control scheme is developed by utilizing radial basis function neural networks (RBF NNs) to deal with nonlinear system uncertainties. Specifically, the Galerkin method is first employed to derive a reduced-order ordinary differential equation (ODE) model to approximate the original PDE system. Based on this, an adaptive tracking control scheme is developed based on the singular perturbation theory and Lyapunov stability theory. With the control scheme implemented on the original PDE system, the system output can be guaranteed to track a prescribed reference trajectory with desired system stability and tracking accuracy. Simulation study on a representative transport-reaction process is conducted to demonstrate the effectiveness of the proposed approach.**

I. INTRODUCTION

Distributed parameter systems (DPSs) are dynamical systems with inputs, outputs and process parameters varying temporally and spatially [1]. They are usually modeled by partial differential equations (PDEs) [1]. Many thermal process, fluid flow process, biological process, and convection-diffusion reaction process are typical examples of such systems [2].

Modeling and control of DPSs are the most important and challenging problems in the system and control community. One of the technical difficulties lies in how to deal with the infinite-dimensional nature of DPSs. Extensive research has been carried out for this challenging problem over the past decades (see, e.g., [3], [4], [5], [6]), which in general can be categorized into two types: late lumping approach and early lumping approach. In the late lumping approach, the control design is performed using the infinite-dimensional process model, and the obtained infinite-dimensional controller is then lumped for implementation purpose. Some successes have been seen along this direction, e.g., [3], [4], which however are mainly limited to linear systems. For the DPSs with more complicated nonlinear system dynamics, most of existing methods are developed based on the early lumping approach, in which the controller design is based on a finite-dimensional approximate model capturing the essential dynamics of the original process model. A typical example is the control design of a class of dissipative DPSs modeled by parabolic PDEs [7]. The eigenspectrum of their spatial differential operator can be partitioned into a finite-dimensional slow one and an infinite-dimensional stable fast complement. With this feature, the Galerkin method can be employed to derive a reduced-order ODE (slow) system capturing the dominant dynamics of the original PDE system, which can be used to facilitate the subsequent control design. Considerable research works have been reported using this technique, e.g., [5], [6], [7]. However, these schemes virtually require the system model to be known or at least the structure to be known, which could not be applicable for the DPSs with uncertain unstructured nonlinear dynamics.

For systems with unknown nonlinearities, the past decades have witnessed tremendous progress in the research of control design by using neural networks (NNs) together with adaptive control techniques, e.g., [8], [9]. Particularly, with the approximation capability and online learning ability of NN [10], the effect of system uncertain nonlinearities can be properly dealt with, and the system stability, control accuracy and robustness performance of the resulting control system can be guaranteed based on the Lyapunov stability theory [11]. Along this direction, some research efforts have been devoted to the control problem of DPSs. For example, in [12], radial basis function neural networks (RBF NNs) were employed to deal with the system unknown nonlinearity and persistent bounded disturbances, and an adaptive NN-based control was proposed with guaranteed $\mathcal{L}_\infty$ gain performance. [13] developed a Galerkin-NN-based control scheme guaranteeing an upper bound of a quadratic cost function. In [14], critic NN was utilized to attenuate the effect of system disturbance for improving the system's robustness property. However, most of these methods are focused on the stabilization control problem. For the tracking control problem, which is a more important problem in practical applications, only a few research results have been obtained in [3], [15], [16], [17]. Specifically, the tracking problem of linear DPSs was investigated in [3], [15], [16]; while for the nonlinear case, there is only a study in [17] employing a fuzzy interpolation approach to deal with the system uncertain nonlinearity. This method required a high-order ordinary differential equation (ODE) system for good approximation, which could be computationally expensive for real-time implementation. It is therefore of interest to develop a new adaptive NN-based reference-tracking control

This work was supported in part by the National Science Foundation under grant CMMI-1929729.

J. Zhang and C. Yuan are both with the Department of Mechanical, Industrial and Systems Engineering, University of Rhode Island, Kingston, RI 02881, USA (e-mail: jingting_zhang@uri.edu; cyuan@uri.edu)

Y. Gu is with the Department of Mechanical Engineering, University of Massachusetts Lowell, Lowell, MA 01854, USA (e-mail: yan_gu@uml.edu)

P. Stegagno is with the Department of Electrical, Computer and Biomedical Engineering, University of Rhode Island, Kingston, RI 02881, USA (e-mail: pstegagno@uri.edu)

W. Zeng is with the School of Mechanical and Electrical Engineering, Longyan University, Longyan, China (e-mail: zw0597@126.com)

scheme for DPSs with nonlinear uncertainties.

In this paper, we consider the reference-tracking control problem of DPSs modeled by a class of parabolic PDEs with uncertain nonlinear dynamics, aiming to drive the system output to track a prescribed reference trajectory. An adaptive tracking control scheme will be developed by utilizing the RBF NN to deal with the effect of system uncertain nonlinear dynamics. Specifically, the Galerkin method is first employed to derive a reduced-order ODE system to approximate the original PDE system. Based on this ODE system, an adaptive NN-based tracking control scheme is subsequently developed by employing the singular perturbation theory and Lyapunov stability theory [11]. With this controller implemented on the original PDE system, it is verified through rigorous analysis that the output of the PDE system can be guaranteed to track the output of the reference model with desired system stability and tracking accuracy. Simulation study of a representative transport-reaction process is conducted to demonstrate the effectiveness of the proposed approach.

Notation. $\mathbb{R}$, $\mathbb{R}_+$ and $\mathbb{N}_+$ denote, respectively, the set of real numbers, the set of positive real numbers and the set of positive integers; $\mathbb{R}^{m\times n}$ denotes the set of $m \times n$ real matrices; $\mathbb{R}^n$ denotes the set of $n \times 1$ real column vectors; $|\cdot|$ is the absolute value of a real number; $\|\cdot\|$ is the 2-norm of a vector or a matrix.

II. Preliminaries and Problem Fomulation

A. Preliminaries

The RBF networks can be described by $f_{nn}(Z) = \sum_{i=1}^{N_n} w_i s_i(Z) = W^T S(Z)$ [10], where $Z \in \Omega_Z \subset \mathbb{R}^q$ is the input vector, $W = [w_1, \cdots, w_{N_n}]^T \in \mathbb{R}^{N_n}$ is the weight vector, N_n is the NN node number, and $S(Z) = [s_1(\|Z-\varsigma_1\|), \cdots, s_{N_n}(\|Z-\varsigma_{N_n}\|)]^T$, with $s_i(\cdot)$ being a radial basis function, and ς_i $(i = 1, 2, \cdots, N_n)$ being distinct points in state space. The Gaussian function $s_i(\|Z-\varsigma_i\|) = \exp[\frac{-(Z-\varsigma_i)^T(Z-\varsigma_i)}{\eta_i^2}]$ is one of the most commonly used radial basis functions, where $\varsigma_i = [\varsigma_{i1}, \varsigma_{i2}, \cdots, \varsigma_{iq}]^T$ is the center of the receptive field and η_i is the width of the receptive field. The Gaussian function belongs to the class of localized RBFs in the sense that $s_i(\|Z-\varsigma_i\|) \to 0$ as $\|Z\| \to \infty$. Note that $S(Z)$ is bounded, and there exists a real constant $S_M \in \mathbb{R}_+$ such that $\|S(Z)\| \leqslant S_M$ [9]. According to [10], for any continuous function $f(Z) : \Omega_Z \to \mathbb{R}$ where $\Omega_Z \subset \mathbb{R}^q$ is a compact set, and for the NN approximator, where the node number N_n is sufficiently large, there exists an ideal constant weight vector W^*, such that for any $\epsilon^* > 0$, $f(Z) = W^{*T} S(Z) + \epsilon$, $\forall Z \in \Omega_Z$, where $|\epsilon| < \epsilon^*$ is the ideal approximation error. The ideal weight vector W^* is an "artificial" quantity required for analysis, and is defined as the value of W that minimizes $|\epsilon|$ for all $Z \in \Omega_Z \subset \mathbb{R}^q$, i.e., $W^* \triangleq arg\min_{W\in\mathbb{R}^{N_n}} \{\sup_{Z\in\Omega_Z} |f(Z) - W^T S(Z)|\}$.

B. Problem Formulation

Consider a class of nonlinear parabolic PDE systems:

$$\frac{\partial x(z,t)}{\partial t} = a_1 \frac{\partial x(z,t)}{\partial z} + a_2 \frac{\partial^2 x(z,t)}{\partial z^2} + f(x) + k_u b(z) u(t), \tag{1}$$

with the system output: $y(t) = \int_{z_1}^{z_2} c(z) x(z,t) dz$, the boundary conditions: $m_i x(z_i,t) + n_i \frac{\partial x}{\partial z}(z_i,t) = d_i$ $(i = 1, 2)$, and initial condition: $x(z,0) = x_0(z)$, where $x(z,t) \in \mathbb{R}$ is system state, $u(t) \in \mathbb{R}^p$ is system input, $y(t) \in \mathbb{R}$ is system output, $z \in [z_1, z_2]$ is the spatial coordinate, $t \in [0,\infty)$ is the time, $f(x) \in \mathbb{R}$ is an unknown nonlinear function satisfying $f(0) = 0$ which represents system uncertain dynamics, $b(z) = [b_1(z), \cdots, b_p(z)]$ is a known smooth vector function, where $b_i(z)$ $(i = 1, \cdots, p)$ describes how the control action $u_i(t)$ is distributed in the interval $[z_1, z_2]$, and $c(z)$ is a known function that is determined by the desired performance specifications in the process domain $[z_1, z_2]$. $\frac{\partial x}{\partial z}$ and $\frac{\partial^2 x}{\partial z^2}$ are the first- and second-order spatial derivatives of $x(z,t)$, respectively. $k_u, a_1, a_2, m_1, n_1, m_2, n_2, d_1, d_2$ are known constants. Assume that the state $x(z,t)$ is measurable at all locations $z \in [z_1, z_2]$ for all time $t \in [0,\infty)$.

Assumption 1: The function $f(x)$ in (1) is locally Lipschitz continuous, i.e., for each compact subset $\mathbb{D}$ of $\mathbb{R}$, there exists a positive constant κ, such that $|f(x_1) - f(x_2)| \leq \kappa |x_1 - x_2|$ for any $x_1, x_2 \in \mathbb{D}$.

In the following, the Galerkin method [7] will be employed to obtain a low-order ODE system to approximate the PDE system (1). Specifically, denote $\mathcal{H}$ as a Hilbert space of 1-D functions defined on $[z_1, z_2]$ that satisfies the boundary conditions given in (1), with inner product and norm: $\langle \omega_1, \omega_2 \rangle = \int_{z_1}^{z_2} \omega_1(z)\omega_2(z) dz$, $\|\omega_1\|_2 = \langle \omega_1, \omega_1 \rangle^{\frac{1}{2}}$, where $\omega_1(z)$, $\omega_2(z)$ are two elements of $\mathcal{H}$. Consider the system (1), define the spatial operation $\mathcal{A}$ as: $\mathcal{A}x = a_1 \frac{\partial x}{\partial z} + a_2 \frac{\partial^2 x}{\partial z^2}$, $x \in D(\mathcal{A}) := \{x \in \mathcal{H} \,|\, \mathcal{A}x \in \mathcal{H}, m_i x(z_i,t) + n_i \frac{\partial x}{\partial z}(z_i,t) = d_i, i = 1, 2\}$. For this operator, the eigenvalue problem is defined as $\mathcal{A}\phi_j = \lambda_j \phi_j$ $(j = 1, 2, \cdots, \infty)$, where λ_j denotes an eigenvalue, and ϕ_j denotes an eigenfunction. The eigenspectrum of $\mathcal{A}$, denoted by $\sigma(\mathcal{A})$, is defined as the set of all eigenvalues of $\mathcal{A}$, i.e., $\sigma(\mathcal{A}) = \{\lambda_1, \lambda_2, \cdots, \lambda_\infty\}$. For highly-dissipative PDE systems, the eigenspectrum of $\mathcal{A}$ can be partitioned into a finite-dimensional part consisting of m $(m \in \mathbb{N}_+)$ slow eigenvalues and a stable infinite-dimensional complement containing the remaining fast eigenvalues, and the separation between the slow and fast eigenvalues of $\mathcal{A}$ is large. These properties can be satisfied by the majority of diffusion-convection-reaction processes [1], and are stated precisely in the following assumption.

Assumption 2: (i) $\text{Re}\{\lambda_1\} \geq \text{Re}\{\lambda_2\} \geq \cdots \geq \text{Re}\{\lambda_j\} \geq \cdots$, where $\text{Re}\{\lambda_j\}$ denotes the real part of λ_j; (ii) $\sigma(\mathcal{A})$ can be partitioned as $\sigma(\mathcal{A}) = \sigma_s(\mathcal{A}) + \sigma_f(\mathcal{A})$, where $\sigma_s(\mathcal{A})$ consists of the first m number of eigenvalues, that is, $\sigma_s(\mathcal{A}) = \{\lambda_1, \lambda_2, \cdots, \lambda_m\}$, and $|\frac{\text{Re}\{\lambda_1\}}{\text{Re}\{\lambda_m\}}| = O(1)$; (iii) $\text{Re}\{\lambda_{m+1}\} < 0$ and $|\frac{\text{Re}\{\lambda_m\}}{\text{Re}\{\lambda_{m+1}\}}| = O(\varpi)$, where $\varpi := |\frac{\text{Re}\{\lambda_1\}}{\text{Re}\{\lambda_{m+1}\}}| < 1$ is a small positive constant.

Based on this assumption, consider the decomposition $\mathcal{H} = \mathcal{H}_s \oplus \mathcal{H}_f$, in which $\mathcal{H}_s = \text{span}\{\phi_1, \phi_2, \cdots, \phi_m\}$ denotes the finite dimensional space spanned by the slow eigenfunctions corresponding to $\sigma_s(\mathcal{A})$, and $\mathcal{H}_f = \text{span}\{\phi_{m+1}, \phi_{m+2}, \cdots, \phi_\infty\}$ denotes the infinite dimensional complement one spanned by the fast eigenfunctions corresponding to $\sigma_f(\mathcal{A})$. Under such decomposition, by the separation of time and spatial variables, the PDE system (1) can be formulated as the following infinite-dimensional nonlinear ODE system:

$$\begin{aligned}\dot{x}_s &= A_s x_s + f_s(x_s, \chi_f) + B_s u, \quad x_s(0) = x_{s_0}, \\ \dot{\chi}_f &= A_f \chi_f + f_f(x_s, \chi_f) + B_f u, \quad \chi_f(0) = \chi_{f_0},\end{aligned} \tag{2}$$

with $y = C_s^T x_s + C_f^T \chi_f$, where $x_s = [x_{s_1}, \cdots, x_{s_m}]^T \in \mathbb{R}^m$, $\chi_f = [\chi_{f_{m+1}}, \cdots, \chi_{f_\infty}]^T \in \mathbb{R}^\infty$, $f_s = \langle \phi_s(z), f(x) \rangle$, $f_f = \langle \phi_f(z), f(x) \rangle$, $B_s = \langle \phi_s(z), k_u b(z) \rangle$, $B_f = \langle \phi_f(z), k_u b(z) \rangle$, $C_s = \langle \phi_s(z), c(z) \rangle$, $C_f = \langle \phi_f(z), c(z) \rangle$, with $\phi_s(z) = [\phi_1(z), \cdots, \phi_m(z)]^T$, $\phi_f(z) = [\phi_{m+1}(z), \cdots, \phi_\infty(z)]^T$. Under Assumption 2, A_s is a diagonal matrix of dimension $m \times m$ of the form $A_s = \text{diag}\{\lambda_1, \cdots, \lambda_m\}$, and A_f is an unbounded differential operator of the form $A_f = \text{diag}\{\lambda_{m+1}, \cdots, \lambda_\infty\}$, which generates a strongly continuous exponentially stable semigroup (following from the part (iii) of Assumption 2 and the selection of $\mathcal{H}_s$ and $\mathcal{H}_f$). Neglecting the fast mode, we can obtain the following finite-dimensional slow system:

$$\dot{x}_s = A_s x_s + f_s(x_s, 0) + B_s u, \quad y = C_s^T x_s. \tag{3}$$

Note that this system can still be not directly applied for control design due to the presence of system uncertainty $f_s(x_s, 0)$. To facilitate the control design, the following assumption is given:

Assumption 3: For the function $f_s(x_s, 0)$ in (3), there exist unknown functions $g(x_s)$, $L(x_s)$ and an unknown matrix $L_m > 0$ satisfying: $f_s(x_s, 0) = L(x_s)x_s + B_s g(x_s)$ and $|L(x_s)| \leq L_m$.

Remark 1: The existence of L_m can be guaranteed under Assumption 1. Specifically, noting that the function $f_s(x_s, 0)$ satisfies $f_s(0, 0) = 0$ and is locally Lischitz continuous under Assumption 1, it leads to that $|f_s(x_s, 0)| \leq L_s |x_s|$ for some L_s. Based on this, we can deduce that there exists a constant matrix L_m and some function $g(x_s)$ satisfying $|f(x_s, 0) - B_s g(x_s)| \leq L_m |x_s|$. Under the above assumption, the model (3) can be rewritten as: $\dot{x}_s = (A_s + L(x_s))x_s + B_s(u + g(x_s))$.

In the next section, an adaptive NN-based control scheme will be proposed based on the reduced-order ODE system (3), aiming to drive the system output y to track a given reference trajectory y_d. To this end, we assume that there exists a known reference model given as:

$$\dot{x}_d = A_d x_d + B_d r_d, \quad y_d = C_d^T x_d, \tag{4}$$

where $x_d \in \mathbb{R}^m$, $r_d \in \mathbb{R}^p$, $A_d \in \mathbb{R}^{m \times m}$, $B_d \in \mathbb{R}^{m \times p}$, $C_d \in \mathbb{R}^{1 \times m}$. In this paper, since B_s and C_s of (3) are both known, without loss of generality, we assume $B_d = B_s$, and $C_d = C_s$. In light of (3)–(4), it is clear that to drive the system output y of (3) to track the reference signal y_d of (4), we can design a control scheme for the system state x_s of (3) to track the reference signal x_d of (4). To facilitate the subsequent control design, the following assumptions are made on the reference model (4).

Assumption 4: For the matrix A_d in (4), there exists a matrix K_s such that: $A_d = A_s + L_m - B_s K_s$, where A_s, B_s are from (3), and L_m is defined in Assumption 3.

Assumption 5: The signals x_d, r_d of the reference model (4) are bounded, i.e., there exist some positive numbers x_{dm}, r_{dm} such that: $|x_d| \leq x_{dm}$, $|r_d| \leq r_{dm}$.

III. Adaptive NN Control Design

Given the reduced-order ODE system (3) and the reference model (4), we first consider an ideal scenario that the function $g(x_s)$ (defined in Assumption 3) and the matrix K_s (defined in Assumption 4) are known, and a stabilizing tracking control strategy u^* can be designed as:

$$u^* = -K(x_s - x_d) + r_d - K_s x_s - g(x_s), \tag{5}$$

with $K \in \mathbb{R}^{p \times m}$ being a control gain matrix to be designed. With this control strategy, from (3)–(4), we can obtain the state tracking error dynamics (with $e := x_s - x_d$):

$$\dot{e} = (A_d - B_s K)e - (L_m - L(x_s))e - (L_m - L(x_s))x_d. \tag{6}$$

Note that $L_m - L(x_s) \geq 0$ and x_d is bounded, the tracking error e can be guaranteed stable as long as the control gain matrix K is designed such that $A_d - B_s K < 0$.

However, note that $g_s(x_s)$ and K_s are unknown, the controller (5) is not implementable. The RBF NNs will be employed to overcome this issue. According to Section II, there exists an ideal constant NN weight $W^* \in \mathbb{R}^{N_n \times p}$ (with N_n denoting the number of NN nodes) such that

$$K_s x_s + g(x_s) = W^{*T} S(x_s) + \epsilon \tag{7}$$

where $S(x_s) : \mathbb{R}^m \to \mathbb{R}^{N_n}$ is a smooth RBF vector and $\epsilon \in \mathbb{R}^p$ is the ideal approximation error satisfying $|\epsilon| < \epsilon^*$ with $\epsilon^* \in \mathbb{R}^p$ being a positive constant that can be made arbitrarily small given a sufficient large number of neurons. Based on this, we propose to design an adaptive NN controller:

$$\begin{aligned}u &= -K(x_s - x_d) + r_d - \hat{W}^T S(x_s) \\ \dot{\hat{W}} &= \Gamma S(x_s)(x_s - x_d)^T P_s B_s - \Gamma \gamma \hat{W},\end{aligned} \tag{8}$$

where K, $\Gamma = \Gamma^T > 0$, $P_s = P_s^T > 0$ are design matrices, $\gamma > 0$ is a design constant of small value, $\hat{W} \in \mathbb{R}^{N_n \times p}$ is the estimate of W^* in (7).

In the following, we will verify that with the controller (8) implemented on the original PDE system (2), desired tracking control performance can be guaranteed. To this end, consider the system (2), by multiplying both sides of the χ_f subsystem with ϖ (defined in Assumption 2), we can obtain a standard singular perturbed (SP) form as follows:

$$\begin{aligned}\dot{x}_s &= A_s x_s + f_s(x_s, \chi_f) + B_s u, \\ \varpi \dot{\chi}_f &= A_{f\varpi} \chi_f + \varpi f_f(x_s, \chi_f) + \varpi B_f u,\end{aligned} \tag{9}$$

where $A_{f\varpi} = \varpi A_f$. According to the singular perturbation theory [11, Chap. 11], by introducing a fast time $\tau = \frac{t}{\varpi}$ and setting $\varpi = 0$, we can obtain the following infinite-dimensional fast subsystem from (9)

$$\frac{d\chi_f}{d\tau} = A_{f\varpi} \chi_f. \tag{10}$$

According to the definition of ϖ and the fact $\text{Re}\{\lambda_{m+1}\} < 0$, we know that the system (10) is globally exponentially stable.

Then, according to [11], there exists a matrix $P_f = P_f^T > 0$ that satisfies the following Lyapunov equation:

$$P_f A_{f\varpi} + A_{f\varpi}^T P_f \le -Q_f, \tag{11}$$

where $Q_f > 0$ is a given matrix.

Theorem 1: Consider the closed-loop system consisting of the plant (2), the reference model (4), and the controller (8). Under the Assumptions 1–5, with initial conditions $x(z,0) \in \Omega_0$ (where Ω_0 is a compact set) and $\hat{W}(0) = 0$, assume that there exists symmetric positive definite matrices $\hat{Q}_s, \hat{P}_s$ and a matrix $\hat{K}$ such that

$$\hat{P}_s A_d^T + A_d \hat{P}_s - (\hat{K}^T B_s^T + B_s \hat{K}) \le -\hat{Q}_s. \tag{12}$$

Then, under the control gain of (8) given as $P_s = \hat{P}_s^{-1}$ and $K = \hat{K} P_s$, we have: there exists a positive real number ϖ^* such that for $\varpi \in (0, \varpi^*]$, (i) all signals in the closed-loop system remain uniformly ultimately bounded (UUB); and (ii) the fast state χ_f and the state tracking error $e = x_s - x_d$, as well as the output tracking error $e_y = y - y_d$, all converge exponentially to a small neighborhood around the origin.

Proof: (i) To prove the first part, from (4), (7)–(9), we obtain the following error dynamics:

$$\begin{aligned}
\dot{e} &= A_c e + \Delta f_s - (L_m - L(x_s))x_s - B_s \tilde{W}^T S(x_s) + B_s \epsilon, \\
\dot{\tilde{W}} &= \dot{\hat{W}} = \Gamma S(x_s) e^T P_s B_s - \Gamma \gamma \hat{W}, \\
\varpi \dot{\chi}_f &= A_{f\varpi} \chi_f + \varpi f_f(x_s, \chi_f) + \varpi B_f u,
\end{aligned} \tag{13}$$

with $\tilde{W} = \hat{W} - W^*$, $A_c = A_d - B_s K$ and $\Delta f_s = f_s(x_s, \chi_f) - f_s(x_s, 0)$. Particularly, note that $f_s(x_s, \chi_f)$ and $f_f(x_s, \chi_f)$ are both Lipschitz continuous under Assumption 1, there exist positive real numbers r_1^*, r_1^*, κ_1, κ_2, and κ_3 such that: for any $\|x_s\| \le r_1^*$ and $\|\chi_f\| \le r_2^*$,

$$\begin{aligned}
&\|\Delta f_s\| = \|f_s(x_s, \chi_f) - f_s(x_s, 0)\| \le \kappa_1 \|\chi_f\|, \\
&\|f_f(x_s, \chi_f)\| \le \kappa_2 \|x_s\| + \kappa_3 \|\chi_f\| = \kappa_2 \|e + x_d\| \\
&\quad + \kappa_3 \|\chi_f\| \le \kappa_2 \|e\| + \kappa_2 \|x_d\| + \kappa_3 \|\chi_f\|.
\end{aligned} \tag{14}$$

Consider the positive definite Lyapunov function candidate for (13) as: $V(e, \chi_f, \tilde{W}) = V_s(e, \tilde{W}) + V_f(\chi_f)$, where $V_s(e, \tilde{W}) = e^T P_s e + tr(\tilde{W}^T \Gamma^{-1} \tilde{W})$ and $V_f(\chi_f) = \chi_f^T P_f \chi_f$. From (12)–(14) and the definitions of $x_s = e + x_d$, $\hat{P}_s = P_s^{-1}$, $\hat{Q}_s = P_s^{-1} Q_s P_s^{-1}$ and $\hat{K} = K P_s^{-1}$, we have

$$\begin{aligned}
\dot{V}_s =& e^T(A_c^T P_s + P_s A_c)e - 2e^T P_s (L_m - L(x_s))(e + x_d) \\
&+ 2e^T P_s \Delta f_s - 2e^T P_s B_s \tilde{W}^T S(x_s) + 2e^T P_s B_s \epsilon \\
&+ 2tr(\tilde{W}^T S(x_s) e^T P_s B_s) - 2tr(\tilde{W}^T \gamma \hat{W}) \\
\le& - e^T Q_s e - 2e^T P_s (L_m - L(x_s)) x_d + 2e^T P_s \Delta f_s \\
&+ 2e^T P_s B_s \epsilon - 2\gamma \left\|\tilde{W}\right\|^2 - 2\gamma tr(\tilde{W}^T W^*) \\
\le& - \mu_1 \|e\|^2 + 2\mu_9 \|e\| \|x_d\| + 2\mu_3 \kappa_1 \|e\| \|\chi_f\| \\
&+ 2\mu_2 \|e\| \|\epsilon\| - 2\gamma \left\|\tilde{W}\right\|^2 + 2\gamma \left\|\tilde{W}\right\| \|W^*\|,
\end{aligned} \tag{15}$$

where $L_m - L(x_s) \ge 0$ under Assumption 3, and $\mu_1 = \underline{\sigma}(Q_s)$, $\mu_2 = \bar{\sigma}(P_s B_s)$, $\mu_3 = \bar{\sigma}(P_s)$, $\mu_9 = \bar{\sigma}(P_s(L_m - L(x_s)))$, with $\underline{\sigma}(\cdot)$, $\bar{\sigma}(\cdot)$ denoting the minimum and maximum eigenvalues of $(\cdot)$, respectively.

Then, from (8), (11)–(14), $\dot{V}_f$ can be derived as:

$$\begin{aligned}
\dot{V}_f \le& - \frac{1}{\varpi} \chi_f^T Q_f \chi_f + 2\chi_f^T P_f f_f - 2\chi_f^T P_f B_f K e \\
&+ 2\chi_f^T P_f B_f r_d - 2\chi_f^T P_f B_f \hat{W}^T S(x_s) \\
\le& - \frac{\mu_5}{\varpi} \|\chi_f\|^2 + 2\mu_8 \|\chi_f\| \|e\| + 2\mu_7 \|\chi_f\| \|r_d\| \\
&+ 2\mu_6 \|\chi_f\| (\kappa_2 \|e\| + \kappa_2 \|x_d\| + \kappa_3 \|\chi_f\|) \\
&+ 2\mu_7 S_M \|\chi_f\| (\left\|\tilde{W}\right\| + \|W^*\|)
\end{aligned} \tag{16}$$

where $\mu_5 = \underline{\sigma}(Q_f)$, $\mu_6 = \bar{\sigma}(P_f)$, $\mu_7 = \bar{\sigma}(P_f B_f)$, $\mu_8 = \bar{\sigma}(P_f B_f K)$, S_M is the upper bound of $\|S(x_s)\|$.

Based on (15)–(16), by completing the squares and taking the upper bounds of $|\epsilon| < \epsilon^*$ in (7) and $|r_d| < r_{dm}$, $|x_d| < x_{dm}$ under Assumption 5, we obtain $\dot{V} = \dot{V}_s + \dot{V}_f$ as:

$$\begin{aligned}
&\dot{V} \le -\mu_1 \|e\|^2 - 2\gamma \left\|\tilde{W}\right\|^2 - (\frac{\mu_5}{\varpi} - 2\mu_6 \kappa_3) \|\chi_f\|^2 \\
&+ 2(\mu_3 \kappa_1 + \mu_6 \kappa_2 + \mu_8) \|e\| \|\chi_f\| + 2\mu_9 \|e\| \|x_d\| \\
&+ 2\mu_2 \|e\| \|\epsilon\| + 2\mu_7 \|\chi_f\| \|r_d\| + 2\mu_6 \kappa_2 \|\chi_f\| \|x_d\| \\
&+ 2\mu_7 S_M \|\chi_f\| (\left\|\tilde{W}\right\| + \|W^*\|) + 2\gamma \left\|\tilde{W}\right\| \|W^*\| \\
&< - \frac{\mu_1}{2} \|e\|^2 - \frac{\gamma}{2} \left\|\tilde{W}\right\|^2 - (\frac{\mu_5}{\varpi} - \rho_1) \|\chi_f\|^2 + \frac{\gamma}{2} \|r_{dm}\|^2 \\
&+ \frac{6\mu_2^2}{\mu_1} \|\epsilon^*\|^2 + (\frac{\gamma}{2} + \frac{6\mu_9^2}{\mu_1}) \|x_{dm}\|^2 + \frac{3\gamma}{2} \|W^*\|^2
\end{aligned} \tag{17}$$

with $\rho_1 = 2\mu_6 \kappa_3 + \frac{4(\mu_7 S_M)^2 + 2(\mu_6 \kappa_2)^2 + 2\mu_7^2}{\gamma} + \frac{6(\mu_3 \kappa_1 + \mu_6 \kappa_2 + \mu_8)^2}{\mu_1}$. Defining $\varpi_1 := \frac{\mu_5}{\rho_1}$, if $\varpi \in (0, \varpi_1]$, we have $\dot{V} < 0$ whenever:

$$\begin{aligned}
\|e\| \ge& \frac{2\sqrt{3}\,|\mu_2|}{\mu_1} \|\epsilon^*\| + \sqrt{\frac{\gamma}{\mu_1}} \|r_{dm}\| \\
&+ \sqrt{\frac{\gamma}{\mu_1} + \frac{12\mu_9^2}{\mu_1^2}} \|x_{dm}\| + \sqrt{\frac{3\gamma}{\mu_1}} \|W^*\|, \\
\left\|\tilde{W}\right\| \ge& \sqrt{\frac{12\mu_2^2}{\gamma \mu_1}} \|\epsilon^*\| + \|r_{dm}\| + \sqrt{1 + \frac{12\mu_9^2}{\gamma \mu_1}} \|x_{dm}\| \\
&+ \sqrt{3} \|W^*\|, \\
\|\chi_f\| \ge& \sqrt{\frac{\varpi}{\mu_5 - \varpi \rho_1}} (\sqrt{\frac{6\mu_2^2}{\mu_1}} \|\epsilon^*\| + \sqrt{\frac{\gamma}{2}} \|r_{dm}\| \\
&+ \sqrt{\frac{\gamma}{2} + \frac{6\mu_9^2}{\mu_1}} \|x_{dm}\| + \sqrt{\frac{3\gamma}{2}} \|W^*\|).
\end{aligned} \tag{18}$$

This leads to the uniform boundedness of e, $\tilde{W}$ and χ_f. Noting that $e = x_s - x_d$ and x_d are bounded, x_s is bounded and $S(x_s)$ is also bounded. This enables that the control signal u of (8) (where r_d is bounded) is bounded. Thus, all the signals in the closed-loop system remain UUB.

(ii) To prove the second part, we first consider the Lyapunov function for the dynamics of χ_f in (13), i.e., $V_f(\chi_f) = \chi_f^T P_f \chi_f$. Following a similar line of (16), $\dot{V}_f$ is derived as:

$$\begin{aligned}
\dot{V}_f \le& - \frac{\mu_5}{\varpi} \|\chi_f\|^2 + 2\mu_7 \|\chi_f\| \|u\| \\
&+ 2\mu_6 \|\chi_f\| (\kappa_2 \|e\| + \kappa_2 \|x_d\| + \kappa_3 \|\chi_f\|) \\
\le& - (\frac{\mu_5}{2\varpi} - 2\mu_6 \kappa_3) \|\chi_f\|^2 + \frac{2\varpi}{\mu_5} \bar{\delta}_1^2,
\end{aligned} \tag{19}$$

where $\bar{\delta}_1$ represents the upper bound of $\mu_6\kappa_2\|e\| + 2\mu_6\kappa_4\|x_d\| + \mu_7\|u\|$ (following the fact that the signals e, x_d and u are bounded). Then, defining $\varpi_2 := \frac{\mu_5}{4\mu_6\kappa_3}$ and $\varpi^* = \min(\varpi_1, \varpi_2)$, if $\varpi \in (0, \varpi^*]$, the inequality (19) can be derived as:

$$\dot{V}_f \leq -(\frac{\mu_5}{2\mu_6\varpi} - 2\kappa_3)V_f + \frac{2\varpi}{\mu_5}\bar{\delta}_1^2, \tag{20}$$

where $V_f = \chi_f^T P_f \chi_f \leq \mu_6\|\chi_f\|^2$ with $\mu_6 = \bar{\sigma}(P_f)$. Denote $\rho_2 = \frac{\mu_5}{2\mu_6} - 2\varpi\kappa_3$, the inequality (20) yields:

$$0 \leq V_f \leq V_f(0)e^{-\frac{\rho_2}{\varpi}t} + \frac{2\varpi^2}{\mu_5\rho_2}\bar{\delta}_1^2. \tag{21}$$

Denoting $\mu_8 = \underline{\sigma}(P_f)$, we have:

$$\|\chi_f\| \leq \sqrt{\frac{\mu_6}{\mu_8}}\|\chi_f(0)\| e^{-\frac{\rho_2}{2\varpi}t} + \varpi\bar{\delta}_1\sqrt{\frac{2}{\mu_5\mu_8\rho_2}} := \chi_f^*. \tag{22}$$

This verifies that the fast state χ_f will exponentially converge to a small neighborhood of the origin. The size of such a neighborhood (represented by $\varpi\bar{\delta}_1\sqrt{\frac{2}{\mu_5\mu_8\rho_2}}$) can be made arbitrarily small can be made arbitrarily large by appropriately selecting the parameter ϖ in a sufficiently small value.

Further consider the Lyapunov function for the dynamics of e in (13), i.e., $V_e(e) = e^T P_s e$. Following a similar line of (15)–(17), the derivative of V_e can be obtained as:

$$\begin{aligned}\dot{V}_e \leq & -\mu_1\|e\|^2 + 2\mu_3\kappa_1\|e\|\|\chi_f\| + 2\mu_9\|e\|\|x_d\| \\ & + 2\mu_2\|e\|\|\epsilon\| + 2\mu_2\|e\|\left\|\tilde{W}^T S(x_s)\right\| \\ \leq & -\frac{\mu_1}{2}\|e\|^2 + \frac{8(\mu_3\kappa_1)^2}{\mu_1}\chi_f^{*2} + \frac{8\mu_9^2}{\mu_1}\|x_{dm}\|^2 \\ & + \frac{8\mu_2^2}{\mu_1}\|\epsilon^*\|^2 + \frac{8\mu_2^2}{\mu_1}\tilde{W}^{*2}S_M^2 \\ \leq & -\frac{\mu_1}{2}\|e\|^2 + \frac{8\bar{\delta}_2}{\mu_1} \leq -\frac{\mu_1}{2\mu_3}V_e + \frac{8\bar{\delta}_2}{\mu_1}\end{aligned} \tag{23}$$

with χ_f^* defined in (22), $\bar{\delta}_2 := (\mu_3\kappa_1)^2\chi_f^{*2} + \mu_9^2\|x_{dm}\|^2 + \mu_2^2\|\epsilon^*\|^2 + \mu_2^2\tilde{W}^{*2}S_M^2$ and $\mu_3 = \bar{\sigma}(P_s)$. This leads to:

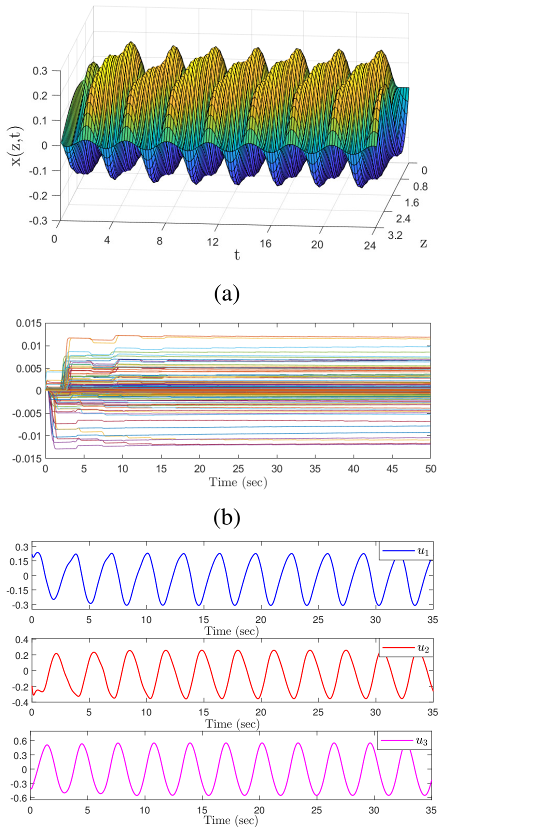
$$0 \leq V_e(t) < V_e(0)e^{-\frac{\mu_1}{2\mu_3}t} + \frac{16\mu_3\bar{\delta}_2}{\mu_1^2}. \tag{24}$$

Based on this, denoting $\mu_4 = \underline{\sigma}(P_s)$, we have:

$$\|e\| \leq \sqrt{\frac{\mu_3}{\mu_4}}\|e(0)\| e^{-\frac{\mu_1}{4\mu_3}t} + \frac{4}{\mu_1}\sqrt{\frac{\mu_3\bar{\delta}_2}{\mu_4}}. \tag{25}$$

This verifies that the tracking error e will converge exponentially to a small vicinity around the origin within a finite time. The size of such a neighborhood is represented by $\frac{4}{\mu_1}\sqrt{\frac{\mu_3\bar{\delta}_2}{\mu_4}}$, which can be made arbitrarily small by appropriately selecting the control gain matrix $Q_s = P_s\hat{Q}_s P_s$ such that the associated parameter $\mu_1 = \underline{\sigma}(Q_s)$ is sufficiently large.

Consequently, from (22) and (25), it is verified that both the state χ_f and the tracking error e will exponentially converge to a small neighborhood around the origin. Based on this, for the output tracking error $e_y = y - y_d$, from (2), (4), (22) and (25), we have: $e_y = C_s e + C_f\chi_f$ will also exponentially converge to a small neighborhood around the origin. This ends the proof. ■



(a)

(b)

(c)

Fig. 1: System responses: (a) system state $x(z,t)$; (b) NN weight $\hat{W}$; and (c) control signal $u = [u_1, u_2, u_3]^T$.

IV. Simulation Studies

To demonstrate effectiveness of the proposed control scheme, we consider the control problem of the temperature profile of a catalytic rod [12], [13]. The mathematical model which describes the spatio-temporal evolution of the rod temperature can be represented in the form of (1), i.e., $a_1\frac{\partial x}{\partial z} + a_2\frac{\partial^2 x}{\partial z^2} + f(x) + k_u b(z)u(t) = \frac{\partial^2 x}{\partial z^2} + \beta_T(e^{-\frac{\gamma}{1+x}} - e^{-\gamma}) + \beta_u(b(z)u(t) - x)$, subject to the Dirichlet boundary conditions and initial condition: $x(0,t) = 0$, $x(\pi,t) = 0$, and $x(z,0) = 0.1\sin(z)$, in which $x(z,t)$ denotes the rod temperature; $u(t) \in \mathbb{R}^3$ is the manipulated input; $b(z) = [h(z)-h(z-\frac{\pi}{3}), h(z-\frac{\pi}{3})-h(z-\frac{2\pi}{3}), h(z-\frac{2\pi}{3})-h(z-\pi)]$ is control distribution function with $h(\cdot)$ being a Heaviside function; $c(z) = h(z-\frac{\pi}{2}) - h(z-\pi)$; $\beta_T = 50$ denotes a heat of reaction, $\gamma = 4$ denotes an activation energy, $\beta_u = 2$ denotes a heat transfer coefficient.

The eigenvalue problem for the spatial differential operator of the process (1): $\mathcal{A}x = \frac{\partial^2 x}{\partial z^2}$, $x \in D(\mathcal{A}) := \{x \in \mathcal{H} \,|\, x(0,t) = 0, x(\pi,t) = 0\}$ can be solved analytically and its solution is of the form: $\lambda_j = -j^2$, $\phi_j(z) = \sqrt{\frac{2}{\pi}}\sin(jz)$ $(j = 1, 2, \cdots, \infty)$. We consider the first $m = 3$ eigenvalues as the dominant ones. As a result, we can obtain a 3-D ODE system in the form of (3), with $A_s = \mathrm{diag}(-1,-4,-9)$, $B_s = \sqrt{\frac{2}{\pi}}[1,2,1;\frac{3}{2},0,-\frac{3}{2};\frac{4}{3},-\frac{4}{3},\frac{4}{3}]$, $C_s = \sqrt{\frac{2}{\pi}}[1;-1;\frac{1}{3}]$ and $f_{s_i}(x_s,0) = -\beta_u x_{s_i} + \beta_T \int_0^{\pi} \phi_i(z)(e^{-\frac{\gamma}{1+\sum_{i=1}^{3} x_{s_i}\phi_i(z)}} - e^{-\gamma})dz$ $(i = 1,2,3)$. The reference model is given in the form of (4) with $A_d = \mathrm{diag}(-2,-2,-2)$, and $r_d(t) = [0.3\sin(2t+\frac{\pi}{2}); 0.3\sin(2t+\pi); 0.3\sin(2t+\frac{3\pi}{2})]$.

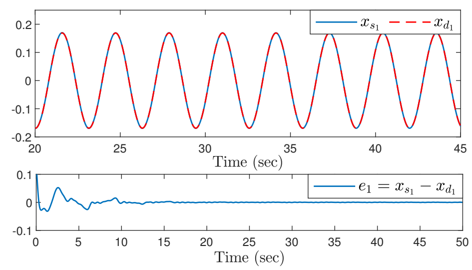


Fig. 2: State tracking performance of system (3) to reference model (4) using controller (8): $x_{s_1} \to x_{d_1}$.

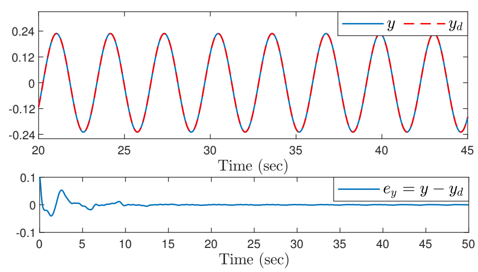


Fig. 3: Output tracking performance of y in system (1) to y_d in reference model (4).

Based on the system (3) and the reference model (4), the adaptive NN controller can be designed according to (8). Specifically, the RBF NNs $\hat{W}^T S(x_s)$ are constructed in a regular lattice, with nodes $N_n = 8 \times 8 \times 8$, the centers evenly spaced on $[-0.3, 0.3] \times [-0.3, 0.3] \times [-0.3, 0.3]$ and the widths $\eta_i = 0.05$ $(i = 1, 2, \cdots, 2197)$. The design parameters are obtained by solving (12) as $P_s = 1$, $K = [0.6262, 1.2524, 0.9414; 1.2543, 0.0019, -0.9428; 0.6281, -1.2505, 0.9357]$, which leads to $Q_s = \mathrm{diag}(10, 10, 10)$, $\Gamma = 0.5$ and $\gamma = 0.005$, respectively. The initial conditions are set as $x_d(0) = [0, 0, 0]^T$ and $\hat{W}(0) = 0$. With such a system setup, considering the system (1) and the reference model (4), the control performance of the designed controller (8) are plotted in Figs. 1-3. Specifically, Fig. 1 implies that all signals in the closed-loop system, including the system state $x(z, t)$, the NN weight $\hat{W}$ and the control signal u, are stable. Fig. 2 illustrates accurate tracking performance of the state x_{s_1} of (3) to reference signal x_{d_1} of (4); similar results are obtained for x_{s_2} and x_{s_3}, thus associated plots are omitted here. Accurate tracking performance of output y in (1) to reference signal y_d in (4) is achieved in Fig. 3.

V. Conclusions

In this paper, an adaptive RBF NN-based reference-tracking control scheme has been proposed for parabolic PDE systems with uncertain nonlinear dynamics. Specifically, the Galerkin method was first employed to derive a reduced-order ODE system to capture the dominant dynamics of the PDE system. Based on this, an adaptive tracking control scheme can be designed by employing the singular perturbation theory and Lyapunov stability theory. With this controller implemented on the original PDE system, it has been also verified rigorously that the system output can be guaranteed to track a prescribed reference trajectory with desired system stability and tracking accuracy.

References

[1] G. Montaseri and M. J. Yazdanpanah, "Predictive control of uncertain nonlinear parabolic pde systems using a galerkin/neural-network-based model," *Communications in Nonlinear Science and Numerical Simulation*, vol. 17, no. 1, pp. 388–404, 2012.

[2] B. Hernandez-Morales and A. Mitchell, "Review of mathematical models of fluid flow, heat transfer, and mass transfer in electroslag remelting process," *Ironmaking & steelmaking*, vol. 26, no. 6, pp. 423–438, 1999.

[3] T. Meurer and A. Kugi, "Tracking control for boundary controlled parabolic pdes with varying parameters: Combining backstepping and differential flatness," *Automatica*, vol. 45, no. 5, pp. 1182–1194, 2009.

[4] Q. Fu, W.-G. Gu, P.-P. Gu, and J.-R. Wu, "Iterative learning control for a class of mixed hyperbolic-parabolic distributed parameter systems," *International Journal of Control, Automation and Systems*, vol. 14, no. 6, pp. 1455–1463, 2016.

[5] S. Dubljevic, P. Mhaskar, N. H. El-Farra, and P. D. Christofides, "Predictive control of transport-reaction processes," *Computers & chemical engineering*, vol. 29, no. 11-12, pp. 2335–2345, 2005.

[6] N. H. El-Farra, A. Armaou, and P. D. Christofides, "Analysis and control of parabolic pde systems with input constraints," *Automatica*, vol. 39, no. 4, pp. 715–725, 2003.

[7] P. D. Christofides and P. Daoutidis, "Finite-dimensional control of parabolic pde systems using approximate inertial manifolds," *Journal of mathematical analysis and applications*, vol. 216, no. 2, pp. 398–420, 1997.

[8] J. Zhang, C. Yuan, C. Wang, P. Stegagno, and W. Zeng, "Composite adaptive nn learning and control for discrete-time nonlinear uncertain systems in normal form," *Neurocomputing*, vol. 390, pp. 168–184, 2020.

[9] C. Wang and D. J. Hill, *Deterministic learning theory for identification, recognition, and control*. Boca Raton, FL, USA: CRC Press, 2009.

[10] M. POEWELL, *The Theory of Radial Basis Function Approximation*. Oxford: Clarendon Press, 1992.

[11] H. K. Khalil, *Nonlinear Systems*. Upper Saddle River, NJ: Prentice-Hall, 2002.

[12] H.-N. Wu and H.-X. Li, "Adaptive neural control design for nonlinear distributed parameter systems with persistent bounded disturbances," *IEEE transactions on neural networks*, vol. 20, no. 10, pp. 1630–1644, 2009.

[13] ——, "A galerkin/neural-network-based design of guaranteed cost control for nonlinear distributed parameter systems," *IEEE transactions on neural networks*, vol. 19, no. 5, pp. 795–807, 2008.

[14] Y. Luo, Q. Sun, H. Zhang, and L. Cui, "Adaptive critic design-based robust neural network control for nonlinear distributed parameter systems with unknown dynamics," *Neurocomputing*, vol. 148, pp. 200–208, 2015.

[15] C. Bymes, I. G. Laukó, D. S. Gilliam, and V. I. Shubov, "Output regulation for linear distributed parameter systems," *IEEE Transactions on Automatic Control*, vol. 45, no. 12, pp. 2236–2252, 2000.

[16] M. J. Balas and S. A. Frost, "Robust adaptive model tracking for distributed parameter control of linear infinite-dimensional systems in hilbert space," *IEEE/CAA Journal of Automatica Sinica*, vol. 1, no. 3, pp. 294–301, 2014.

[17] Y.-T. Chang and B.-S. Chen, "A fuzzy approach for robust reference-tracking-control design of nonlinear distributed parameter time-delayed systems and its application," *IEEE Transactions on fuzzy systems*, vol. 18, no. 6, pp. 1041–1057, 2010.