Supplementary Material for the Paper Titled "Time-Varying Foot Placement Control for Humanoid Walking on Swaying Rigid Surface"

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This document presents the supplementary material for the paper titled "Time-Varying Foot Placement Control for Humanoid Walking on Swaying Rigid Surface."

The equations and references in this supplementary document are labeled in continuation of those in the main paper. The mathematical symbols are defined consistently with those in the main paper.

A. Algorithm 1

In Algorithm 1, the value of a variable $(\cdot)$ at t is denoted by $(\cdot)_t$.

Algorithm 1 Pseudocode for updating the Bézier polynomial coefficients at time step t based on the ALIP-DRS planner

while True **do**
  At time step t, obtain the current generalized coordinates $\mathbf{q}_t$ of the full-order robot.
  if s=0 **then**
    Assign the current swing-foot position in the x- and y-directions to $\alpha_{5,0}$ and $\alpha_{6,0}$:
    $\alpha_{5,0} \leftarrow x_{Sw}(\mathbf{q}_t)$, $\alpha_{6,0} \leftarrow y_{Sw}(\mathbf{q}_t)$.
  end
  if $0 \le s \le 1$ **then**
    Convert $\mathbf{q}_t$ into the current ALIP-DRS state:
    $\mathbf{x}_t \leftarrow \mathbf{x}(\mathbf{q}_t)$, $\mathbf{y}_t \leftarrow \mathbf{y}(\mathbf{q}_t)$.
    Compute the pre-impact angular momentum $L_{y,S}(T_k^-)$ and $L_{x,S}(T_k^-)$ using (14) and (27), based on $\mathbf{x}_t$, $\mathbf{y}_t$, and known DRS motion.
    Compute $(x_{SwC}(T_k^-), y_{SwC}(T_k^-))$ using (19) and ((28).
    Assign the planned swing-foot landing locations $(x_{SwC}(T_k^-), y_{SwC}(T_k^-))$ to the appropriate Bézier coefficients of ϕ_5 and ϕ_6:
    $\alpha_{5,M} \leftarrow -x_{SwC}(T_k^-)$, $\alpha_{6,M} \leftarrow -y_{SwC}(T_k^-)$.
    Assign $\alpha_{5,1}$,...,$\alpha_{5,M-1}$ as the linear interpolation between $\alpha_{5,0}$ and $\alpha_{5,M}$.
    Assign $\alpha_{6,1}$,...,$\alpha_{6,M-1}$ as the linear interpolation between $\alpha_{6,0}$ and $\alpha_{6,M}$.
  else
    No update on $\alpha_{5,0}$,...,$\alpha_{5,M}$ and $\alpha_{6,0}$,...,$\alpha_{6,M}$.
  end
end

B. Proof of Proposition 1

Proof: The proof of Proposition 1 is built on the stability theory of linear time-invariant impulsive systems in the form of $\begin{cases} \dot{\mathbf{z}} = \mathbf{A}\mathbf{z}^-, & t \neq \tau_k^- \\ \Delta\mathbf{z} = \mathbf{B}_k\mathbf{z}^-, & t = \tau_k^- \end{cases}$ with $k \in \mathbb{N}$, which have the following characteristics: (i) the matrix $\mathbf{A}$ is constant; (ii) the matrix $\mathbf{B}_k$ is constant within $\tau_k^+ \le t \le \tau_{k+1}^-$, and satisfies $\det(\mathbf{I}+\mathbf{B}_k) \neq 0$; and (iii) there exists $q \in \mathbb{N}$ and $T > 0$ such that $\mathbf{B}_{k+q} = \mathbf{B}_k, \tau_{k+q} = \tau_k + T$ with $T > 0$. It can be verified that the homogeneous system in (22) satisfies conditions (i)-(iii). Conditions (i) and (ii) are met because $\mathbf{A} = \mathbf{A}_x$ and $\mathbf{B}_k = \mathbf{B}_x$ are both constant square matrices, T_k is essentially τ_k, and by the definition of $\mathbf{B}_x$ we know $\mathbf{I}+\mathbf{B}_x$ is nonsingular. Also, from $T_{sys} = N_1 T_{step}$ and $T_{step} = T_{k+1} - T_k$, we know $T_{k+N_1} = T_k + T_{sys}$, confirming (22) meets condition (iii).

Thanks to these properties, we can assess the stability of the homogeneous system (22) based on the stability theorem of general linear time-invariant impulsive systems given in Theorem 3.5 of [43]. By that theorem, the homogeneous system (22) is exponentially stable if and only if all the eigenvalues of its monodromy matrix are less than 1 in modulus. Because the eigenvalues $\lambda_{x,1}$ and $\lambda_{x,2}$ of (22) are strictly less than one in modulus under the proposed discrete control law in (20), as proven in Sec. III.B.3), (22) is exponentially stable. □

C. Expression of the Low-Level IO-QP Controller

1) Cost function: To minimize the control command magnitude and reduce the deviation of the IO-QP from the ideal input-output linearizing control law, the cost function is chosen as $\boldsymbol{\tau}_u^\top \boldsymbol{\tau}_u + p\boldsymbol{\delta}_{QP}^\top \boldsymbol{\delta}_{QP}$, where the penalty weight p can be adjusted to indirectly minimize the norm of the slack variables $\boldsymbol{\delta}_{QP}$.

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2) Constraints: To enforce the proposed input-output linearizing control law augmented with the slack variables $\boldsymbol{\delta}_{QP}$, the IO-QP control law in (33) is treated as an equality constraint.

Furthermore, the inequality constraint that the control input $\boldsymbol{\tau}_u$ must respect is: $\mathbf{u}_{min} \leq \boldsymbol{\tau}_u \leq \mathbf{u}_{max}$, where $\mathbf{u}_{max}$ and $\mathbf{u}_{min}$ represent the upper and lower limits of $\boldsymbol{\tau}_u$, respectively.

Also, combining (31) and (32) yields the following equality that agrees with the continuous-phase dynamics model in (31) under the holonomic constraint in (32):

$$\mathbf{N}_2(\mathbf{q}, \dot{\mathbf{q}}) = \mathbf{0}, \tag{37}$$

where $\mathbf{N}_2 = \mathbf{J}_c \mathbf{M}^{-1}(\mathbf{B}\boldsymbol{\tau}_u + \mathbf{J}_c^\intercal \mathbf{f}_c - \mathbf{c}) + \dot{\mathbf{J}}_c \dot{\mathbf{q}} - \ddot{\mathbf{p}}_c(t)$.

Lastly, the friction cone constraints is introduced as:

$$\mathbf{S}_{FC}\mathbf{f}_c \in \mathcal{FC},$$

where $\mathbf{S}_{FC}$ is a constant selection matrix such that $\mathbf{S}_{FC}\mathbf{f}_c$ is the ground-contact force. Here, $\mathcal{FC}$ represents the linearized friction cone.

3) Optimization variables: Based on the proposed cost function and constraints, the optimization variables are chosen as: (i) the force vector $\mathbf{f}_c$ associated with the holonomic constraints in (32); (ii) the joint torque vector $\boldsymbol{\tau}_u$; and (iii) the slack variables $\boldsymbol{\delta}_{QP}$.

4) QP formulation: In summary, the proposed IO-QP control is given by:

$$\begin{aligned} \min_{\boldsymbol{\tau}_u, \boldsymbol{\delta}_{QP}, \mathbf{f}_c} \quad & \boldsymbol{\tau}_u^\intercal \boldsymbol{\tau}_u + p\boldsymbol{\delta}_{QP}^\intercal \boldsymbol{\delta}_{QP} \\ \text{s.t.} \quad & \boldsymbol{\tau}_u = \mathbf{N}_1 + \boldsymbol{\delta}_{QP}, \\ & \mathbf{N}_2 = \mathbf{0}, \\ & \mathbf{S}_{FC}\mathbf{f}_c \in \mathcal{FC}, \\ & \boldsymbol{\tau}_u \geq \mathbf{u}_{min}, \quad \boldsymbol{\tau}_u \leq \mathbf{u}_{max}. \end{aligned} \tag{38}$$

D. Inverse Kinematics Approach

Due to the presence of springs on Digit (Fig. 7), there is a notable mismatch between the model and the actual robot dynamics. Thus, this study also considers an inverse kinematics approach [37] as a candidate low-level controller.

The closed-form inverse kinematics approach is independent of a robot's dynamics model, and thus can be more robust than the IO-QP controller in practice, even in the presence of Digit's springs. As the original inverse kinematics approach [37] is designed for robots with actuated ankles, we adapt it to walking robots with passive ankles as follows.

Let $\mathbf{q}_a \in \mathbb{R}^{18}$ denote the angles of all actively actuated joints. Then $\mathbf{q}_a = \mathbf{B}_a \mathbf{q}$ with $\mathbf{B}_a$ a constant selection matrix.

The inverse kinematics controller treats the velocities of the actuated joints, $\dot{\mathbf{q}}_a$, as control inputs, which can be directly set to any arbitrary values within the joint limits to command $\mathbf{q}_a$. The objective of the controller is to enforce the output-function system to be stable by following the first-order linear system, $\dot{\mathbf{h}}(\mathbf{q}, \dot{\mathbf{q}}) + \boldsymbol{\kappa}\mathbf{h}(\mathbf{q}) = \mathbf{0}$, where $\boldsymbol{\kappa} \in \mathbb{R}^{18\times 18}$ is a positive-definite control gain matrix.

By the definition of the output function, we know $\dot{\mathbf{h}} = \dot{\mathbf{h}}_a - \dot{\mathbf{h}}_d$, and consequently this first-order system can be rewritten as:

$$\mathbf{J}_{h_a}(\mathbf{q})\dot{\mathbf{q}} = \dot{\mathbf{h}}_d - \boldsymbol{\kappa}\mathbf{h}(\mathbf{q}), \tag{39}$$

where $\mathbf{J}_{h_a}(\mathbf{q}) := \frac{\partial \mathbf{h}_a(\mathbf{q})}{\partial \mathbf{q}} \in \mathbb{R}^{18\times 36}$. Since $\mathbf{J}_{h_a}$ is not invertible, we cannot directly use this equation to solve for the reference velocities $\dot{\mathbf{q}}$ (which contains the actuated joints $\dot{\mathbf{q}}_a$).

The non-invertibility of $\mathbf{J}_{h_a}$ is essentially due to the underactuation induced by Digit's passive support-toe joints as well as the presence of the 16 holonomic constraints in Digit's kinematic structure. Thus, to solve for $\dot{\mathbf{q}}$, we consider the passive support-toe velocities $\dot{\mathbf{q}}_{Toe} \in \mathbb{R}^2$ while explicitly expressing the holonomic constraints using $\dot{\mathbf{q}}$.

We map $\dot{\mathbf{q}}$ into $\dot{\mathbf{q}}_{Toe}$, by using $\mathbf{S}_{Toe}\dot{\mathbf{q}} = \dot{\mathbf{q}}_{Toe}$, where $\mathbf{S}_{Toe} \in \mathbb{R}^{2\times 36}$ is a known constant selection matrix. Meanwhile, to respect the 16 holonomic constraints, $\dot{\mathbf{q}}$ needs to satisfy $\mathbf{J}_c(\mathbf{q})\dot{\mathbf{q}} = \dot{\mathbf{p}}_c(t)$, which corresponds to (32). Combining these two equations with (39) gives:

$$\mathbf{J}_{IK}(\mathbf{q})\dot{\mathbf{q}} = \mathbf{v}_{IK}(t, \mathbf{q}, \dot{\mathbf{q}}) := \begin{bmatrix} \dot{\mathbf{h}}_d - \boldsymbol{\kappa}\mathbf{h}(\mathbf{q}) \\ \dot{\mathbf{p}}_c(t) \\ \dot{\mathbf{q}}_{Toe} \end{bmatrix}, \tag{40}$$

where $\mathbf{J}_{IK}(\mathbf{q}) := [\mathbf{J}_{h_a}^\intercal(\mathbf{q}), \mathbf{J}_c^\intercal(\mathbf{q}), \mathbf{S}_{Toe}^\intercal]^\intercal \in \mathbb{R}^{36\times 36}$. Note that $\mathbf{J}_c(\mathbf{q})$ and $\mathbf{S}_{Toe}$ both have full row rank on Q because the kinematic relationships they represent are independent from each other. Then, $\mathbf{J}_{IK}$ is full-rank on Q if the control variables $\mathbf{h}_a$ are chosen to be independent from each other and from the kinematic relationships underlying $\mathbf{J}_c(\mathbf{q})$ and $\mathbf{S}_{Toe}$.

Thus, the velocity command $\dot{\mathbf{q}}_a$ is given by:

$$\dot{\mathbf{q}}_a = \mathbf{B}_a \mathbf{J}_{IK}^{-1}(\mathbf{q}) \mathbf{v}_{IK}(t, \mathbf{q}, \dot{\mathbf{q}}). \tag{41}$$

Since the actual robot is directly actuated by joint torques instead of joint velocities, we construct a PD joint-torque controller as inspired by [45], which treats the joint velocity command in (41) as the desired velocities denoted as $\dot{\mathbf{q}}_{a,d}$. Then, from (41), we have $\dot{\mathbf{q}}_{a,d} = \mathbf{B}_a \mathbf{J}_{IK}^{-1}(\mathbf{q}) \mathbf{v}_{IK}(t, \mathbf{q}, \dot{\mathbf{q}})$.

Thus, the PD joint-torque controller is designed as:

$$\boldsymbol{\tau}_u = -\bar{\mathbf{K}}_p(\mathbf{q}_a - \mathbf{q}_{a,d}) - \bar{\mathbf{K}}_d(\dot{\mathbf{q}}_a - \dot{\mathbf{q}}_{a,d}), \tag{42}$$

where $\bar{\mathbf{K}}_p, \bar{\mathbf{K}}_d \in \mathbb{R}^{18\times 18}$ are PD matrices to be appropriately tuned. This controller is experimentally implemented on the physical Digit robot, as described in Sec.VII.

E. Proof of Theorem 2

Proof: We begin the proof by first analyzing the evolution of the Lyapunov function candidate $V(\mathbf{X})$ for the $(k+1)^{th}$ walking step on $t \in (T_k, T_{k+1}]$, which comprises one continuous phase and one switching event.

The continuous-phase boundedness of V_h is given in Proposition 2. Meanwhile, since the continuous-phase uncertainty $\mathbf{d}_{A_\eta}$ in the unactuated dynamics is locally bounded and the matrix $\mathbf{A}_\eta$ is constant and finite, we know that there exist positive real constants L_1, L_2, and r_1 such that

$$V_\eta|_{k+1}^- \leq L_1 V_\eta|_k^+ + L_2 \tag{43}$$

holds for any initial state satisfying $\mathbf{X}(0) \in B_{r_1}$.

From Proposition 2 and (43), we know the following inequality holds for any $\mathbf{X}(0) \in B_{r_1}(\mathbf{0})$:

$$\begin{aligned} V|_{k+1}^{-} &= \beta_h V_h|_{k+1}^{-} + \beta_\eta V_\eta|_{k+1}^{-} \\ &\leq \beta_h e^{-c_{h3}(T_{k+1}-T_k)} V_h|_k^{+} + \beta_h c_{h4} + \beta_\eta L_1 V_\eta|_k^{+} + \beta_\eta L_2 \\ &\leq L_v V|_k^{+} + \beta_h c_{h4} + \beta_\eta L_2, \end{aligned} \tag{44}$$

where $L_v := \max(e^{-c_{h3}(T_{k+1}-T_k)}, L_1)$. This inequality verifies that the stability condition (S1) is met.

Since the nonlinear reset map $\mathbf{\Delta}_h$ is continuously differentiable in t and $\mathbf{X}$ [33], we know $\mathbf{\Delta}_h$ is locally Liptchiz in t and $\mathbf{X}$ [46]. Then, by applying the triangle inequality to the reset map $\mathbf{\Delta}_h$ [45], we know there exist positive real scalars L_h, L_η, ε_Δ, and r_Δ such that

$$\left\| \mathbf{x}_h|_{k+1}^{+} \right\|^2 \leq L_h \left\| \mathbf{x}_h|_{k+1}^{-} \right\|^2 + L_\eta \left\| \mathbf{x}_\eta|_{k+1}^{-} \right\|^2 + \varepsilon_\Delta \tag{45}$$

holds for a bounded initial state $\mathbf{X}(0) \in B_{r_\Delta}(\mathbf{0})$.

From Propositions 2 and 3 and (43)-(45), we have:

$$V_h|_{k+1}^{+} \leq \tfrac{L_h c_{h2}}{c_{h1}} e^{-c_{h3}(T_{k+1}-T_k)} V_h|_k^{+} + \tfrac{L_\eta L_1 c_{h2}}{c_{\eta 1}} V_\eta|_k^{+} + \varepsilon_h, \tag{46}$$

where $\varepsilon_h := \varepsilon_\Delta c_{h2} + \frac{L_\eta L_2 c_{h2}}{c_{\eta 1}} + \frac{L_h c_{h4} c_{h2}}{c_{h1}}$.

Then, from the definition of V, Proposition 3, and (46), the step-to-step boundedness of V is given by:

$$V|_{k+1}^{+} \leq c_{v1} V|_k^{+} + c_{v2}, \tag{47}$$

where $c_{v1} := \max(\frac{L_h c_{h2}}{c_{h1}} e^{-c_{h3}(T_{k+1}-T_k)}, c_{\eta 3} + \frac{L_1 L_\eta \beta_h c_{h2}}{\beta_\eta c_{\eta 1}})$ and $c_{v2} := \beta_h \varepsilon_h + \beta_\eta c_{\eta 4}$.

Since β_h and β_η can be chosen arbitrarily, we know c_{v2} can be set arbitrarily small, along with $\frac{L_1 L_\eta \beta_h c_{h2}}{\beta_\eta c_{\eta 1}}$. Meanwhile, we know $c_{\eta 3} < 1$. Thus, we can choose β_h and β_η such that $c_{\eta 3} + \frac{L_1 L_\eta \beta_h c_{h2}}{\beta_\eta c_{\eta 1}}$ is less than one. Meanwhile, by properly choosing the PD gains in (38), the continuous-phase convergence rate c_{h3} can be sufficiently large such that $\frac{L_h c_{h2}}{c_{h1}} e^{-c_{h3}(T_{k+1}-T_k)}$ can be any positive number less than one. Thus, $c_{v1} < 1$ is ensured.

Since both c_{v1} and c_{v2} can be any arbitrarily small positive number with $c_{v1} < 1$, $V|_{k+1}^{+} \leq V|_k^{+}$ is guaranteed for any $k \in \mathbb{N}$ and $\mathbf{X}(0) \in B_{r_v}(\mathbf{0})$ with $r_v = \min(r_d, r_\eta, r_\Delta, r_1, r_h)$. Thus, condition (S2) is met, which completes the proof. □

F. Settings of Bézier Polynomials in Simulations

In MATLAB simulations, the Bézier polynomials defining the desired full-dimensional trajectories $\mathbf{h}_d$ are chosen as follows:

- ϕ_1: Set to H so as to enforce the CoM height of the ALIP-DRS model on the full-order Digit model.
- ϕ_2, ϕ_3, and ϕ_4: Set to 0 for a constant trunk orientation.
- ϕ_5 and ϕ_6: Set according to Algorithm 1.
- ϕ_7: Set with Bézier coefficients $[0, 0.02, 0.07, 0.15, 0.07, 0.02, 0]^\intercal$ to minimize the desired swing-foot height and reduce swing-foot motion.
- ϕ_8, ϕ_9, and ϕ_{10}: Set to 0 to ensure a proper swing-foot orientation at touchdowns.
- ϕ_{11} to ϕ_{18}: Set to 0 to keep the arms static.

G. Determining Upper Bounds of Model Uncertainties

From the closed-loop model in (35), we can express the uncertainty terms as: $\mathbf{d}_{A_h} = \dot{\mathbf{x}}_h - \mathbf{A}_h \mathbf{x}_h$, $\mathbf{d}_{A_\eta} = \dot{\mathbf{x}}_\eta - \mathbf{A}_\eta \mathbf{x}_\eta$, and $\mathbf{d}_{\Delta_\eta} = \mathbf{x}_\eta^{+} - \mathbf{\Delta}_\eta \mathbf{x}_\eta^{-}$. Then, given the test data (e.g., generalized coordinates and velocities, ALIP-DRS state trajectories, contact timing, and desired trajectories), we can use these expressions to compute the uncertainty terms. The maximum of those values can be used as a reference to approximate the associated upper bounds L_{A_h}, L_{A_η}, and L_{Δ_η}.

Following this procedure and using the test data from Case A, the upper bounds L_{A_h}, L_{A_η}, and L_{Δ_η} are computed as $L_{A_h} = 7$, $L_{A_\eta} = 40$, and $L_{\Delta_\eta} = 0.05$.

H. Uncertainty Analysis of ALIP-DRS

In real-world applications, foot placement errors and state estimation errors are inevitable due to various uncertainties such as model inaccuracy and hardware imperfections. This subsection analyzes the effects of foot placement and state estimation errors on the state convergence based on the proposed ALIP-DRS model. This analysis is an extension of the uncertainty analysis in [35] from the time-invariant LIP model to the time-varying ALIP-DRS with non-matching DRS and gait periods. The outcome of this analysis can be used to provide insights into the maximum uncertainty the proposed foot placement controller can tolerate while maintaining the closed-loop stability of the ALIP-DRS.

Since the ALIP-DRS dynamics in the sagittal and lateral planes share the same structure, the uncertainty analysis in this section focuses on the sagittal plane without loss of generality and for brevity.

1) Solution to the nominal open-loop ALIP-DRS model: The proposed robustness analysis begins with deriving the solution (i.e., state evolution) of the nominal open-loop ALIP-DRS model.

Recall the open-loop ALIP-DRS given in (13) is:

$$\begin{cases} \dot{\mathbf{x}} = \mathbf{A}_x \mathbf{x} + \mathbf{f}_x(t), & \text{if } t \neq T_k^{-}; \\ \Delta \mathbf{x} = \mathbf{b}_x u_x, & \text{if } t = T_k^{-}, \end{cases}$$

where the expressions of $\mathbf{A}_x$, $\mathbf{f}_x$, and $\mathbf{b}_x$ are given in Sec. II of the main paper. The model does not consider uncertainties caused by foot placement and state estimation errors.

The solution of the continuous-time portion of this ALIP-DRS model (i.e., $\dot{\mathbf{x}} = \mathbf{A}_x \mathbf{x} + \mathbf{f}_x(t)$) can be obtained through integration on $t \in [T_k^{+}, T_{k+1}^{-}]$ for any $k \in \mathbb{N}$. Also, the solution to the discrete part (i.e., $\Delta \mathbf{x} = \mathbf{b}_x u_x$) at $t = T_{k+1}^{+}$ can be derived based on $\Delta \mathbf{x} = \mathbf{x}(T_{k+1}^{+}) - \mathbf{x}(T_{k+1}^{-})$.

Accordingly, the solutions at $t = T_{k+1}^{-}$ and $t = T_{k+1}^{+}$ are respectively given by:

$$\begin{aligned} \mathbf{x}(T_{k+1}^{-}) &= e^{\mathbf{A}_x T_{step}} \mathbf{x}(T_k^{+}) + \mathbf{V}_x(T_k^{+}, T_{k+1}^{-}) \\ \mathbf{x}(T_{k+1}^{+}) &= \mathbf{x}(T_{k+1}^{-}) + \mathbf{b}_x u_x, \end{aligned} \tag{48}$$

where the expression of $\mathbf{V}_x$ is given in Sec. III.A.

2) Solution to the closed-loop ALIP-DRS model under foot placement and state estimation errors: Based on the solution of the nominal open-loop ALIP-DRS in (48) and the proposed foot placement control law u_x given in (20), we

derive the closed-loop state evolution under foot placement and state estimation errors as explained next.

This robustness analysis considers a foot placement error $\delta_{u,k} \in \mathbb{R}$ and a state estimation error $\boldsymbol{\delta}_{x,k} \in \mathbb{R}^2$ that both induce the implementation inaccuracy of the foot placement control law u_x at the k^{th} foot-touchdown event.

Specifically, under the foot placement error $\delta_{u,k}$ alone, the solution at the touchdown event becomes $\mathbf{x}(T_k^+) = \mathbf{x}(T_k^-) + \mathbf{b}_x(u_x + \delta_{u,k})$. Meanwhile, recall that from Sec. III.A, the proposed foot placement control law satisfies $\mathbf{b}_x u_x = \mathbf{B}_x\mathbf{x} + \mathbf{g}_x(t)$. Then, under the state estimation error $\boldsymbol{\delta}_{x,k}$, $\mathbf{b}_x u_x$ becomes $\mathbf{B}_x(\mathbf{x}+\boldsymbol{\delta}_{x,k}) + \mathbf{g}_x(t)$ at the the k^{th} landing, instead of $\mathbf{B}_x\mathbf{x} + \mathbf{g}_x(t)$.

Then, with the foot placement and state estimation errors incorporated, the solutions of the uncertain closed-loop ALIP-DRS dynamics are given by:

$$\begin{aligned}\mathbf{x}(T_{k+1}^-) =& e^{\mathbf{A}_x T_{step}}\mathbf{x}(T_k^+) + \mathbf{V}_x(T_k^+, T_{k+1}^-)\\ \mathbf{x}(T_{k+1}^+) =& (\mathbf{I}+\mathbf{B}_x)\mathbf{x}(T_{k+1}^-) + \mathbf{g}_x(T_{k+1}^-)\\ &+ \mathbf{b}_x\delta_{u,k+1} + \mathbf{B}_x\boldsymbol{\delta}_{x,k+1}.\end{aligned} \tag{49}$$

3) Closed-loop step-to-step error dynamics under foot placement and state estimation errors: Consider the tracking error $\mathbf{z} := \mathbf{x} - \boldsymbol{\psi}_x$. Note that $\boldsymbol{\psi}_x$ is a solution of the nominal closed-loop ALIP-DRS in (21) under the foot placement control law in (20), as explained in Sec. III.B. Also, $\mathbf{x}$ is a solution to the uncertain closed-loop ALIP-DRS model given in (49).

Then, the solutions of the uncertain closed-loop error dynamics are given by:

$$\begin{aligned}\mathbf{z}(T_{k+1}^-) &= e^{\mathbf{A}_x T_{step}}\mathbf{z}(T_k^+)\\ \mathbf{z}(T_{k+1}^+) &= (\mathbf{I}+\mathbf{B}_x)\mathbf{z}(T_{k+1}^-) + \mathbf{b}_x\delta_{u,k+1} + \mathbf{B}_x\boldsymbol{\delta}_{x,k+1}.\end{aligned} \tag{50}$$

Combining the two solutions in (50) yields the following closed-loop step-to-step error dynamics:

$$\begin{aligned}\mathbf{z}(T_{k+1}^+) &= (\mathbf{I}+\mathbf{B}_x)e^{\mathbf{A}_x T_{step}}\mathbf{z}(T_k^+) + \mathbf{b}_x\delta_{u.k} + \mathbf{B}_x\boldsymbol{\delta}_{x,k}\\ &=: \mathbf{M}_x\mathbf{z}(T_k^+) + \boldsymbol{\delta}_k.\end{aligned} \tag{51}$$

Note that for the nominal case, the closed-loop error system is essentially the homogeneous system in (22), which is why the matrix $\mathbf{M}_x$ here has the same expression as that defined in (23).

Recall that the least period of the solution to the nominal closed-loop ALIP-DRS model is $T_{sys} = N_1 T_{step}$ with $N_1 \in \mathbb{N}$. Then, the complete step-to-step dynamics of the closed-loop error system under uncertainties is:

$$\mathbf{z}(T_{k+N_1}^+) = \mathbf{M}_x^{N_1}\mathbf{z}(T_k^+) + \sum_{i=1}^{N_1}\mathbf{M}_x^{N_1-i}\boldsymbol{\delta}_{k+i-1}. \tag{52}$$

4) Step-to-step evolution of Lyapunov function: From Sec. III.B of the main paper, we know the closed-loop ALIP-DRS system without uncertainty (i.e., (21)) is exponentially stable under the proposed foot placement control law in (20). Thus, for the step-to-step error dynamics in (52) with $\boldsymbol{\delta}_x = \mathbf{0}$ and $\delta_u = 0$, we know there exists a Lyapunov function $V_z = \mathbf{z}^T\mathbf{P}_z\mathbf{z}$ with the positive-definite, symmetric matrix $\mathbf{P}_z$ a solution to the closed-form Lyapunov equation associated with the nominal portion of (52).

Based on the uncertain closed-loop step-to-step dynamics in (52), the change of V_z over an entire least period of $T_{sys} = N_1 T_{step}$ is obtained as:

$$\begin{aligned}&V_z(T_{k+N_1}^+) - V_z(T_k^+)\\ &= \mathbf{z}^\intercal(T_{k+N_1}^+)\mathbf{P}_z\mathbf{z}(T_{k+N_1}^+) - \mathbf{z}^\intercal(T_k^+)\mathbf{P}_z\mathbf{z}(T_k^+)\\ &= (\mathbf{M}_x^{N_1}\mathbf{z}(T_k^+))^\intercal\mathbf{P}_z(\mathbf{M}_x^{N_1}\mathbf{z}(T_k^+)) - \mathbf{z}^\intercal(T_k^+)\mathbf{P}_z\mathbf{z}(T_k^+)\\ &\quad + (\sum_{i=1}^{N_1}\mathbf{M}_x^{N_1-i}\boldsymbol{\delta}_{k+i-1})^\intercal(\mathbf{P}_z + \mathbf{P}_z^\intercal)(\mathbf{M}_x^{N_1}\mathbf{z}(T_k^+))\\ &\quad + (\sum_{i=1}^{N_1}\mathbf{M}_x^{N_1-i}\boldsymbol{\delta}_{k+i-1})^\intercal\mathbf{P}_z(\sum_{i=1}^{N_1}\mathbf{M}_x^{N_1-i}\boldsymbol{\delta}_{k+i-1})\\ &= \mathbf{z}^\intercal(T_k^+)((\mathbf{M}_x^{N_1})^\intercal\mathbf{P}_z\mathbf{M}_x^{N_1} - \mathbf{P}_z)\mathbf{z}(T_k^+)\\ &\quad + 2(\sum_{i=1}^{N_1}\mathbf{M}_x^{N_1-i}\boldsymbol{\delta}_{k+i-1})^\intercal\mathbf{P}_z(\mathbf{M}_x^{N_1}\mathbf{z}(T_k^+))\\ &\quad + (\sum_{i=1}^{N_1}\mathbf{M}_x^{N_1-i}\boldsymbol{\delta}_{k+i-1})^\intercal\mathbf{P}_z(\sum_{i=1}^{N_1}\mathbf{M}_x^{N_1-i}\boldsymbol{\delta}_{k+i-1})\\ &\le -\alpha_1\|\mathbf{z}(T_k^+)\|^2 + 2\alpha_2\|\mathbf{z}(T_k^+)\| + \alpha_3,\end{aligned} \tag{53}$$

where α_1, α_2, and α_3 are positive constants defined as:

$$\begin{aligned}\alpha_1 &= -\lambda_{\max}\big((\mathbf{M}_x^{N_1})^\intercal\mathbf{P}_z\mathbf{M}_x^{N_1} - \mathbf{P}_z\big),\\ \alpha_2 &= \delta_{\max}\lambda_{\max}(\mathbf{P}_z)(\sum_{i=1}^{N_1}\|\mathbf{M}_x\|^{N_1+1-i}),\\ \alpha_3 &= \delta_{\max}^2\lambda_{\max}\big((\sum_{i=1}^{N_1}\mathbf{M}_x^{N_1-i})^\intercal\mathbf{P}_z(\sum_{i=1}^{N_1}\mathbf{M}_x^{N_1-i})\big).\end{aligned} \tag{54}$$

Note that α_1 is ensured to be positive under the proposed foot placement controller in (20). This is because under (20), the nominal closed-loop ALIP-DRS in (21) is exponentially stable (i.e., $\lambda_{\max}\big((\mathbf{M}_x^{N_1})^\intercal\mathbf{P}_z\mathbf{M}_x^{N_1} - \mathbf{P}\big) < 0$).

For the uncertain closed-loop system to be stable,

$$V_z(T_{k+N_1}^+) - V_z(T_k^+) \le 0 \tag{55}$$

should hold for any $k \in \mathbb{N}$. A sufficient condition for the satisfaction of this inequality is $-\alpha_1\|\mathbf{z}(T_k^+)\|^2 + 2\alpha_2\|\mathbf{z}(T_k^+)\| + \alpha_3 \le 0$, which corresponds to the uncertainty ball region defined as:

$$B_r = \{\mathbf{z} \in \mathbb{R}^2 : \|\mathbf{z}\| \le r\}, \tag{56}$$

where $r := \frac{\alpha_2 + \sqrt{\alpha_2^2 + \alpha_1\alpha_3}}{\alpha_1}$.

When the error state $\mathbf{z}(T_k^+)$ ($k \in \mathbb{N}$) is outside B_r, $\mathbf{z}(T_k^+)$ converges to B_r as $k \to \infty$. Thus, a smaller ball (i.e., smaller r) corresponds to a larger asymptotic convergence region in the state space of $\mathbf{z}$. If $r = 0$, the closed-loop error system is globally asymptotically stable.